Recall: If $\vec{c}(t) = (x(t), y(t), z(t))$ is a path then

$f: \mathbb{R}^3 \to \mathbb{R}^1$ (scalar valued)

Path Integral: $$\int_{\vec{c}} f\, ds := \int_a^b f(\vec{c}(t)) \, \|\vec{c}\,'(t)\| \, dt$$

Line Integral: $$\int_{\vec{c}} \vec{F} \cdot d\vec{s} = \int_a^b \vec{F}(\vec{c}(t)) \cdot \vec{c}\,'(t) \, dt$$

vector valued

New notation for the line integral:

$\vec{F} = (P, Q, R)$ & $d\vec{s} = (dx, dy, dz)$

We write $$\int_{\vec{c}} \vec{F} \cdot d\vec{s} = \int_{\vec{c}} P\,dx + Q\,dy + R\,dz$$
$$= \int_a^b \left(P \frac{dx}{dt} + Q \frac{dy}{dt} + R \frac{dz}{dt} \right) dt$$

Warning: This is NOT the sum of 3 integrals. It is just another way to write $\int_{\vec{c}} \vec{F} \cdot d\vec{s}$.

We still need to parametrize and express everything in terms of the parameter.

Example: In the previous example

$\vec{c}(t) = (\cos t, \sin t, t)$, $0 \le t \le 2\pi$

$\vec{F}(x, y, z) = x^2 \vec{i} + y^2 \vec{j} + z \vec{k}$

so $$\int_{\vec{c}} \vec{F} \cdot d\vec{s} = \int_{\vec{c}} x^2\, dx + y^2\, dy + z\, dz$$

$$= \int_0^{2\pi} \left(x^2(t)\frac{dx}{dt} + y^2(t)\frac{dy}{dt} + z(t)\frac{dz}{dt} \right) dt$$

$$= \int_0^{2\pi} \left(\cos^3 t \cdot (-\sin t)\, dt + \sin^2 t \cos t\, dt + t \right) dt$$

$$= 0 + 0 + \frac{4\pi^2}{2} \quad \text{(same as before)}$$

<u>Example</u>:

evaluate $\displaystyle\int_{\vec{C}} x^2\,dx + xy\,dy$

where $\vec{C}(t) = (\underset{x(t)}{t}, \underset{y(t)}{t^2})$, $0 \le t \le 1$

<u>Sol'n</u>:

$$\int_{\vec{C}} x^2\,dx + xy\,dy = \int_0^1 \left(x^2 \overset{dt/dt}{\frac{dx}{dt}} + xy \overset{d(t^2)/dt}{\frac{dy}{dt}} \right) dt$$

$$= \int_0^1 (t^2 \cdot 1 + t^3 \cdot 2t)\, dt$$

$$= \int_0^1 t^2 + 2t^4\, dt = \frac{1}{3} + \frac{2}{5} = \frac{11}{15}.$$

<u>Remark</u>: Line integrals are independent of the parametrization as long as the parametrization is orientation preserving.

ex the curve $y = x^3$ from $(0,0)$ to $(1,1)$
can be parametrized as $\vec{c}(t) = (t, t^3)$ $0 \leq t \leq 1$
or as $\vec{p}(\theta) = (\sin\theta, \sin^3\theta)$, $0 \leq \theta \leq \pi/2$

Let $\vec{F} = x\vec{i} + y\vec{j}$

$$\int_{\vec{c}} \vec{F} \cdot d\vec{s} = \int_0^1 (t\vec{i} + t^2\vec{j}) \cdot \underbrace{\vec{c}\,'(t)}_{\vec{i} + 3t^2\vec{j}}\, dt$$

$$= \int_0^{\pi/2} (\sin\theta\,\vec{i} + \sin^3\theta\,\vec{j}) \cdot \underbrace{\vec{p}\,'(\theta)}_{\cos\theta\,\vec{i} + 3\sin^2\theta\cos\theta\,\vec{j}}\, d\theta$$

~~Exercise~~ check this

— x —

Line integral of gradient field

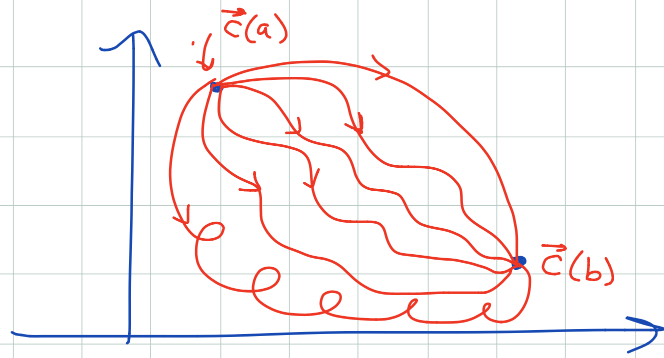
Recall FTC: $\int_a^b f'(t)\,dt = f(b) - f(a)$ FTOLI

Theorem: (Fundamental Thm. of line integrals)

$f : \mathbb{R}^3 \to \mathbb{R}$, differentiable. $c : [a,b] \to \mathbb{R}$ cont's or piecewise

$$\int_{\vec{c}} (\nabla f) \cdot d\vec{s} = f(\vec{c}(b)) - f(\vec{c}(a))$$

In words: If the field is a gradient field, only the end points matter



example: evaluate $\int_{\vec{c}} \nabla f \cdot d\vec{s}$

where $f(x,y,z) = \cos x + \sin y - xyz$ and $\vec{c}$ is a traject. that starts at $(\frac{\pi}{2}, \frac{\pi}{2}, 0)$ and ends at $(\pi, 2\pi, 1)$

Sol'n:

Let $\vec{c}(t)$, $a \le t \le b$, be a path with

$$\vec{c}(a) = (\tfrac{\pi}{2}, \tfrac{\pi}{2}, 0) \;\&\; \vec{c}(b) = (\pi, \tfrac{\pi}{2}, 1)$$

Then:

$$\int_{\vec{c}} \nabla f \cdot d\vec{s} \overset{\text{by FTOLI}}{=} f(\vec{c}(b)) - f(\vec{c}(a))$$

$$= \cos\frac{\pi}{2} + \sin\frac{\pi}{2} - 0 - (\cos\pi + \sin 2\pi - 2\pi^2)$$

$$= 0 + 1 - (-1 - 2\pi^2)$$

$$= 1 + 2\pi^2 .$$

So far: • Integrals of scalar fields along curves (path integrals)

• Integrals of vector fields along curves (line integral)

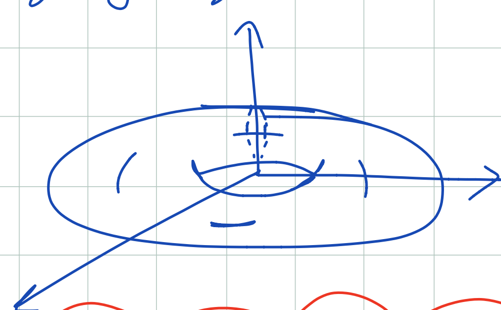
Next: Integrals over surfaces, but first we must learn to parametrize surfaces.

7.3 Parametrized Surfaces

e.g. the graph of a function $f(x,y)$ is a surface.

Can have surfaces that are not the graph of a fct.

e.g. taurus (think surface of a doughnut)

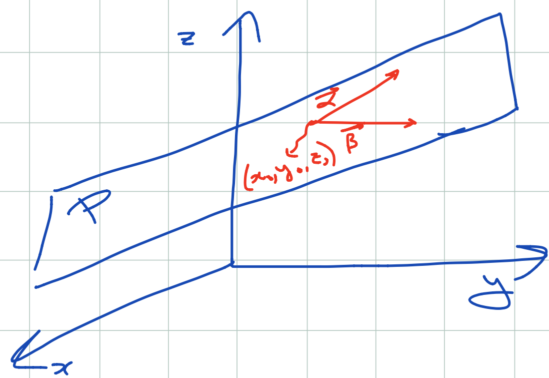


Definition: A parametrization of a surface is a fct $\Phi : D \subset \mathbb{R}^2 \to \mathbb{R}^3$. The surface is $S = \Phi(D)$

so $\Phi(u,v) = (x(u,v), y(u,v), z(u,v))$

(If $x(u,v)$, $y(u,v)$, $z(u,v)$ are differentiable, we call S a diff. surface)

Parametrization of a plane:

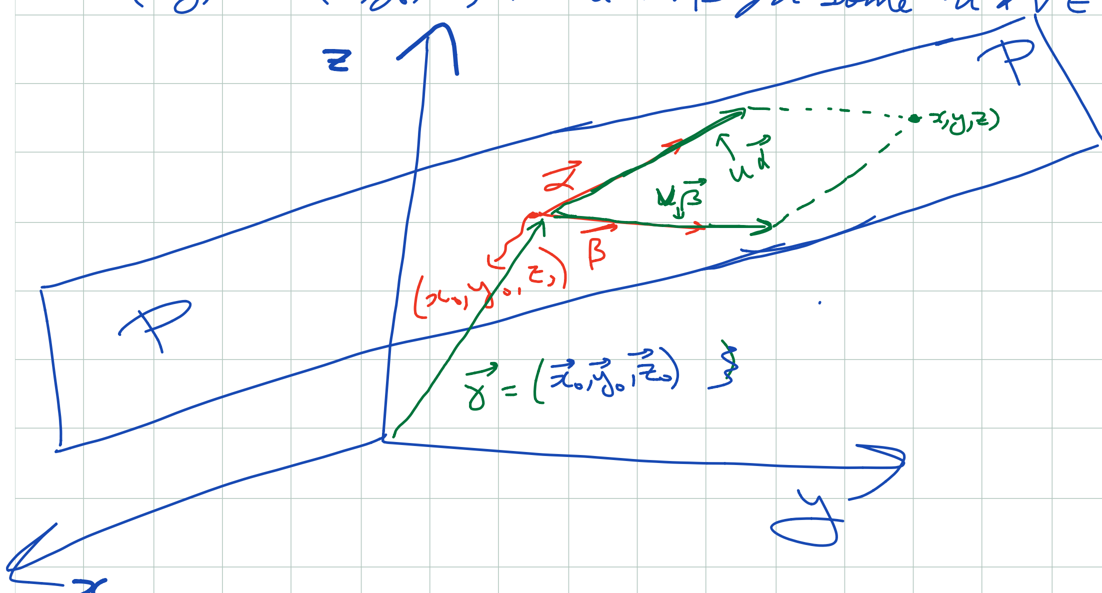


Suppose that $\vec{\alpha}$ & $\vec{\beta}$ are $\parallel$ to P and that $(x_0,y_0,z_0) \in P$

[$\vec{\alpha}$ & $\vec{\beta}$ not $\parallel$ to each other

Then we can write any $(x,y,z) \in P$ as

$(x,y,z) = (x_0,y_0,z_0) + u\vec{\alpha} + v\vec{\beta}$ for some u & $v \in \mathbb{R}$



$$\text{So} \quad \Phi(u,v) = \vec{\alpha}u + \vec{\beta}v + \vec{\gamma}$$

Example: find a parametrization of the plane $x+y+z=1$

the point $(0,0,1)$ is on the plane

& the vectors $(1,-1,0)$ & $(0,1,-1)$ are $\parallel$ to the plane (why) so $\Phi(u,v) = (0,0,1) + (1,-1,0)u + (0,1,-1)v$

Check:
$$\left.\begin{aligned} x(u,v) &= 0 + u \\ y(u,v) &= 0 - u + v \\ z(u,v) &= 1 - v \end{aligned}\right\} \quad x + y + z = 1 \ \checkmark$$

—x—

Example: find a parametrization of the plane $x + y + z = 1$

the point $(0,0,1)$ is on the plane

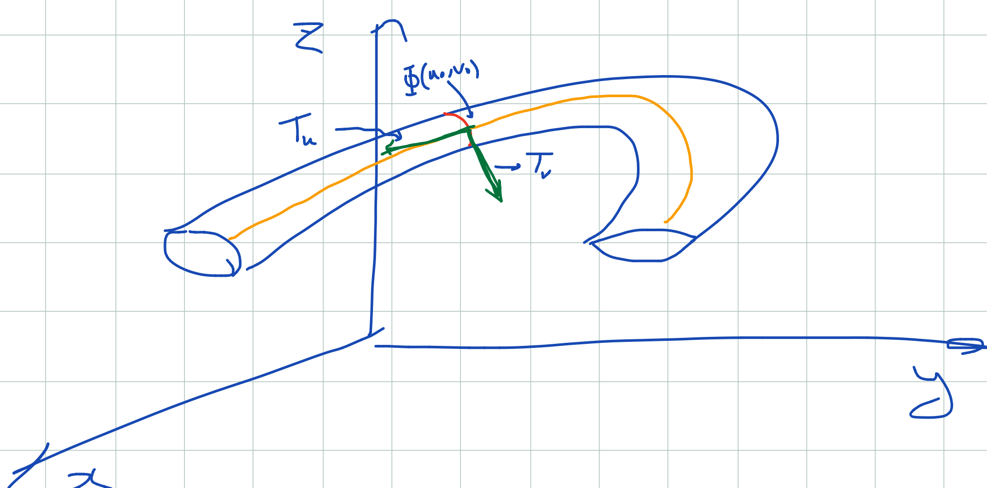
& the vectors $(1,-1,0)$ & $(0,1,-1)$ are $\parallel$ to the plane (why)

so $\Phi(u,v) = (0,0,1) + (1,-1,0)u + (0,1,-1)v$

Check:
$$\left.\begin{aligned} x(u,v) &= 0 + u \\ y(u,v) &= 0 - u + v \\ z(u,v) &= 1 - v \end{aligned}\right\} \quad x + y + z = 1 \ \checkmark$$

—x—

Tangent vectors to parametrized surfaces



- Suppose that $\Phi(u,v)$ is diff. at (u_0,v_0). Fix u_0 & look at the map $\boxed{t \to \Phi(u_0,t)}$ (In otherwords, we have a map $\mathbb{R} \to \mathbb{R}^3$) now, which identifies a curve on the surface (the red one).

- The vector tangent to this curve is given by

$$T_v = \frac{\partial \Phi}{\partial v} = \frac{\partial x}{\partial v}(u_0,v_0)\vec{i} + \frac{\partial y}{\partial v}(u_0,v_0)\vec{j} + \frac{\partial z}{\partial v}(u_0,v_0)\vec{k}$$

it is also tangent to the surface

- Similarly $T_u = \frac{\partial \Phi}{\partial u} = \frac{\partial x}{\partial u}(u_0,v_0)\vec{i} + \frac{\partial y}{\partial u}(u_0,v_0)\vec{j} + \frac{\partial y}{\partial z}(u_0,v_0)\vec{k}$

- T_u & T_v are both tangent to the surface

  - so $T_u \times T_v$ is normal to it (provided $T_u \times T_v \neq 0$)

- So, given $(u_0,v_0) \xrightarrow{\Phi} (x_0,y_0,z_0)$

  to find the eq. of the tang. plane at (u_0,v_0),

  we calculate $\vec{n} = T_u \times T_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \end{vmatrix}_{(u_0,v_0)}$

- Tang. Plane: $\vec{n} \cdot (x-x_0, y-y_0, z-z_0) = 0$

$$\iff n_1(x-x_0) + n_2(y-y_0) + n_3(z-z_0) = 0$$

Example:

Let $\Phi : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ be given by

$\Phi(u,v) = (u\cos v, u\sin v, u^2+v^2)$

Find the tang. plane at $\Phi(1,0)$

Sol'n:

$$\left.\begin{array}{l} T_u = (\cos v, \sin v, 2u) \\ T_v = (-u\sin v, u\cos v, 2v) \end{array}\right\} \Rightarrow \vec{n} = \left|\begin{matrix} \vec{i} & \vec{j} & \vec{k} \\ \cos v & \sin v & 2u \\ -u\sin v & u\cos v & 2v \end{matrix}\right|_{(1,0)} = \left|\begin{matrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & 2 \\ 0 & 1 & 0 \end{matrix}\right|$$

$\vec{n} = \vec{i}(-2) - \vec{j}(\ 0\ ) + \vec{k}(1) = -2\vec{i} + \vec{k}$

$(x_0, y_0, z_0) = (1\cos 0, 1\sin 0, 1^2+0^2) = (1,0,1)$

$\Rightarrow$ eq of tangent plane: $-2(x-1) + 0(y-0) + (1)(z-1) = 0$

$$-2x + z = -1$$

Note: We say that a surface is regular, or smooth at $\Phi(u_0,v_0)$ if $T_u \times T_v \neq 0$ at (u_0,v_0). We say that it is regular, if it is regular at all points $\Phi(u_0,v_0) \in S$.

7.4 Area of a surface

Just as we needed arclength to deal with path integrals, we will need surface area to compute surface integrals.

Definition: The surface area of a parametrized surface

$$A(S) = \iint_D \|T_u \times T_v\| \, du\, dv$$

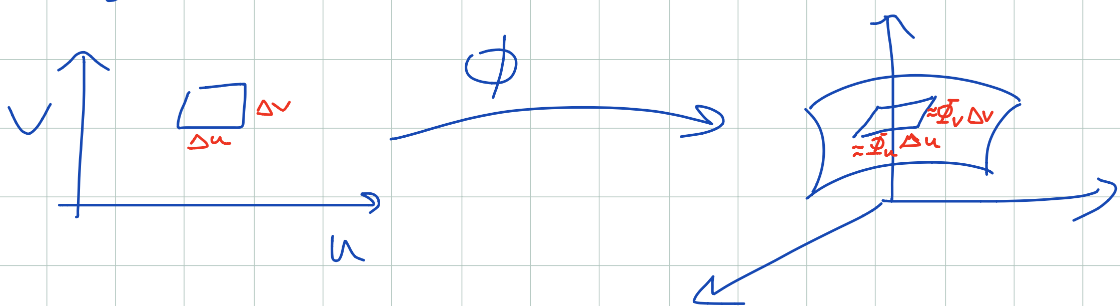
$\Phi(D)$ ↳ elementary region

one-to-one differentiable → regular

Denoting $\dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$, $\|T_u \times T_v\| = \sqrt{\left[\dfrac{\partial(x,y)}{\partial(u,v)}\right]^2 + \left[\dfrac{\partial(y,z)}{\partial(u,v)}\right]^2 + \left[\dfrac{\partial(x,z)}{\partial(u,v)}\right]^2}$

So $$A(S) = \iint_D \sqrt{\left[\frac{\partial(x,y)}{\partial(u,v)}\right]^2 + \left[\frac{\partial(y,z)}{\partial(u,v)}\right]^2 + \left[\frac{\partial(x,z)}{\partial(u,v)}\right]^2} \, du\, dv$$

To see why the def'n of surface area makes sense:



so the area of the image of the small rectangle is

$$\|(\Phi_u \Delta u) \times (\Phi_v \Delta v)\| = \|\Phi_u \times \Phi_v\| \Delta u \Delta v$$

Summing the areas of all such parms as $\Delta u, \Delta v \to 0$ gives $A(S)$.

Example: Surface area of the cone $z = \sqrt{x^2+y^2}$ where $0 \le z \le 1$.

We parametrize $\Phi(r,\theta) = (r\cos\theta, r\sin\theta, r)$, $0 \le r \le 1$, $0 \le \theta \le 2\pi$

So $T_r = \Phi_r = (\cos\theta, \sin\theta, 1)$

$T_\theta = \Phi_\theta = (-r\sin\theta, r\cos\theta, 0)$

& $T_r \times T_\theta = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \cos\theta & \sin\theta & 1 \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix} = \vec{i}(-r\cos\theta) - \vec{j}(r\sin\theta) + \vec{k}(r)$

$\Rightarrow \|T_r \times T_\theta\| = \sqrt{(-r\cos\theta)^2 + (-r\sin\theta)^2 + r^2} = \sqrt{2}\, r$

$\Rightarrow A(\text{cone}) = \int_0^{2\pi}\int_0^1 \sqrt{2}\, r\, dr\, d\theta = \sqrt{2}/2 \cdot 2\pi = \sqrt{2}\,\pi$

Surface area of the graph of a function $f(x,y)$

we can parametrize S by $(x, y, f(x,y))$

or $(u, v, f(u,v))$

$\Rightarrow T_u \times T_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & \partial f/\partial u \\ 0 & 1 & \partial f/\partial v \end{vmatrix} = -\frac{\partial f}{\partial u}\vec{i} - \frac{\partial f}{\partial v}\vec{j} + \vec{k}$

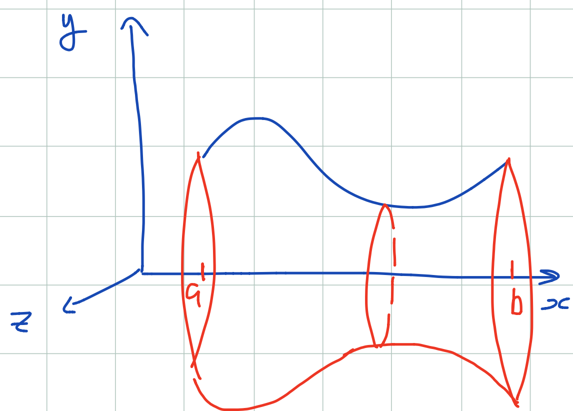
$\Rightarrow A(S) = \iint_D \sqrt{(\partial f/\partial u)^2 + (\partial f/\partial v)^2 + 1}\; du\, dv$

exercise: use this to compute the area of the cone:

$z = \sqrt{x^2+y^2}$

Surfaces of revolution

Suppose S is obtained by rotating the graph of $y = f(x)$, $a \leq x \leq b$ around the x-axis



We can parametrize as S as $(u, f(u)\cos v, f(u)\sin v)$

So $T_u = (1, f'(u)\cos v, f'(u)\sin v)$
$T_v = (0, -f(u)\sin v, f(u)\cos v)$ $\Rightarrow T_u \times T_v = f'(u)f(u)\,\vec{i} + f(u)\cos v\,\vec{j} - f(u)\sin v\,\vec{k}$

$$\Rightarrow A(S) = \int_0^{2\pi}\int_a^b \sqrt{f'(u)^2 + \cos^2 v + \sin^2 v}\,|f(u)|\,du\,dv$$

$$= \int_a^b\int_0^{2\pi} \sqrt{f'(u)^2 + 1}\,|f(u)|\,dv\,du = 2\pi \int_a^b \sqrt{f'(u)^2+1}\,|f(u)|\,du$$

note $A(S) = \int_C 2\pi\,|f(x)|\,dx$ where $c : [a,b] \longrightarrow (t, f(t))$

exercise: use this to compute the area of a cone $z = \sqrt{x^2+y^2}$.